MARKET MODELS

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1. Brief history lesson

In the beginning of time, interest-rate options (caps/floors and swaptions) were valued using Black's model. Black's model was fast. Black's model was well-understood by all, including traders. And Black's model quickly became "industry standard" for these instruments.

Then academicians came. They felt uneasy about models that had not been derived from the "first principles". The no-arbitrage paradigm was applied to interest rate markets. It all culminated with the creation of HJM framework.

To academicians' dismay, traders kept using their beloved Black's model for valuing caps and swaptions. The reasons were pretty clear. The "stochastic drivers" of Black's model (LIBOR and swap rates) were easily observable, and so were their volatilities. On the contrary, the stochastic drivers of HJM models (instantaneous forward rates) were not directly observable, and neither were their volatilities. A Quant equipped with an HJM model was forced to constantly perform translations between observable quantities and his model's input parameters (a process known as calibration). For most of the models, calibration had proved to be too much of a chore, and easily exceeded an average trader's patience and knowledge base.

And then academicians, who by that time moved to Wall street, had a bright idea. They embarked on a quest to create an HJM, no arbitrage model in which caps/swaptions would be valued using Black-like expressions, and where the input parameters would be directly observable on the market.

And so the breed of "market models" was born.

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2. Introduction

Black's model for caps is derived under the assumption that LIBOR rates (corresponding to all caplets in the cap) are lognormally distributed. Likewise, Black's model for swaptions is derived under the assumption that the corresponding swap rate is log-normally distributed.

Any no-arbitrage model in which some of the LIBOR or swap rates are log-normally distributed deserves the name of a "market model".

More broadly, any (no-arbitrage) model where the input parameters are given in terms of "observables" (a subset of LIBOR and swap rates) can also be called a market model.

For the purposes of this lecture we will understand "no arbitrage model" as "HJM model". The class of no-arbitrage models is quite a bit broader than the class of HJM models, but HJM models provide very convenient technical tools (Ito calculus in particular) to be ignored.

Any HJM model is uniquely defined by the volatility structure of instantaneous forward rates (see [L1]). We will build market models by choosing this volatility structure in such a way that observable rates (LIBOR and/or swap) have the dynamics we want. This is our general plan of attack.

Let us recall the notations. Risk-neutral measure is denoted by $\mathbf{Q}$. Brownian motion (under $\mathbf{Q}$) is denoted by W_t (once again we consider one-factor models to save some trees). The model is specified by

$$\begin{aligned} df(t,T) &= -\Sigma(t,T)\,\sigma(t,T)\,dt + \sigma(t,T)\,dW_t, \\ dP(t,T) &= r(t)\,P(t,T)\,dt + P(t,T)\,\Sigma(t,T)\,dW_t, \end{aligned}$$

where

$$\frac{\partial \Sigma(t,T)}{\partial T} = -\sigma(t,T).$$

The volatility structure $\sigma(\cdot,\cdot)$ is as yet unspecified.

3. Simple example

Let us consider a single LIBOR rate, and try to choose volatility structure in such a way that the option on this LIBOR rate (caplet) is priced using Black's formula.

The fixing date of the caplet is set to T, and the tenor to δ. We define the forward LIBOR rate $L(t)$ by

$$L(t) = \frac{P(t,T) - P(t,T+\delta)}{\delta P(t,T+\delta)}.$$

A caplet (call option on L) pays

$$(L(T) - K)^+$$

at time $T+\delta$ (note that the payoff $(L(T)-K)^+$ is determined at time T, so that it is $\mathcal{F}_T$-measurable, yet it is paid later at $T+\delta$). Then ($c(t)$ is the caplet's value at time t)

$$\begin{aligned} c(t) &= B_t \mathbf{E}\left(B_{T+\delta}^{-1} c(T) \middle| \mathcal{F}_t \right) \\ &= B_t \mathbf{E}\left(B_{T+\delta}^{-1} (L(T) - K)^+ \middle| \mathcal{F}_t \right). \end{aligned}$$

Let us change the measure to $T+\delta$-forward. We have,

$$c(t) = P(t, T+\delta)\, \mathbf{E}^{T+\delta}\left((L(T) - K)^+ \middle| \mathcal{F}_t \right).$$

Note that $L(t)$ is the value of a traded asset $(\delta^{-1}(P(t,T) - P(t,T+\delta)))$ divided by the numeraire $P(t,T+\delta)$; hence $L(t)$ is a martingale under $\mathbf{Q}^{T+\delta}$.

Black formula can be derived if we assume that ($\hat{W}$ is a Brownian motion in some measure)

$$\begin{aligned} L(t) &= \exp\left(\lambda \hat{W}_t - \lambda^2 t/2 \right), \\ dL(t) &= \lambda L(t)\, d\hat{W}_t. \end{aligned} \tag{3.1}$$

Note that the process defined by (3.1) is a martingale. So at least our finding is consistent with our goal.

What is the equation for $L(t)$ under $\mathbf{Q}^{T+\delta}$ in our HJM model? For a forward bond $F(t,S,M) = P(t,M)/P(t,S)$ we have (see homework #VP1 or ([L2]))

$$\begin{aligned} dF(t,S,M) &= F(t,S,M)\, \gamma(t,S,M)\, dW_t^S, \\ \gamma(t,S,M) &= \Sigma(t,M) - \Sigma(t,S). \end{aligned}$$

For a LIBOR rate we have (formally $F(t, T+\delta, T)$ is not defined, but all formulas are still valid)

$$L(t) = \delta^{-1}(F(t, T+\delta, T) - 1)$$

and

$$dF(t, T+\delta, T) = F(t, T+\delta, T)\, \gamma(t, T+\delta, T)\, dW_t^{T+\delta},$$

where

$$\gamma(t, T+\delta, T) = \Sigma(t,T) - \Sigma(t, T+\delta),$$

so that

$$\begin{aligned} dL(t) &= \delta^{-1} dF(t, T+\delta, T) \\ &= \delta^{-1} F(t, T+\delta, T) \left[\Sigma(t,T) - \Sigma(t, T+\delta) \right] dW_t^{T+\delta} \\ &= \left(L(t) + \delta^{-1} \right) \left[\Sigma(t,T) - \Sigma(t, T+\delta) \right] dW_t^{T+\delta}. \end{aligned}$$

Then under $\mathbf{Q}^{T+\delta}$ measure,

$$dL(t) = \frac{\delta L(t) + 1}{\delta L(t)} \left[\Sigma(t,T) - \Sigma(t, T+\delta) \right] L(t)\, dW_t^{T+\delta}. \tag{3.2}$$

Let us compare what we want (equation (3.1)) and what we have ((3.2)). We can identify $W_t^{T+\delta}$ and $\hat{W}_t$. Then, as long as we choose $\Sigma(t,T)$ and $\Sigma(t, T+\delta)$ such that

$$\frac{\delta L(t) + 1}{\delta L(t)} \gamma(t, T+\delta, T) = \lambda, \tag{3.3}$$

we are assured that

$$dL(t) = \lambda L(t)\, dW_t^{T+\delta},$$

and the caplet on $L(t)$ is priced using Black's formula (with volatility λ):

$$\begin{aligned} c(0) &= P(0, T+\delta)\, \mathbf{E}^{T+\delta} \left((L(T) - K)^+ \right) \\ &= P(0, T+\delta)\, \mathbf{E}^{T+\delta} \left(\left(L(0)\, e^{\lambda W_T^{T+\delta} - \lambda^2 T/2} - K \right)^+ \right). \end{aligned}$$

So far, we have been presenting the motivation for why market models can be constructed. Now let us present the actual construction of the model in which $L(t)$ has a lognormal distribution. We will pretty much retrace the steps we have outlined above.

1. Specify (observe on the market) a caplet's volatility λ.
2. Specify the dynamics of the LIBOR rate $L(t)$ under $T+\delta$-forward measure $\mathbf{Q}^{T+\delta}$ by
$$dL(t) = \lambda L(t)\, dW_t^{T+\delta},$$
so that
$$L(t) = L(0)\, e^{\lambda W_t^{T+\delta} - \lambda^2 T/2}.$$
3. Define the volatility of the forward bond by
$$\gamma(t, T+\delta, T) = \lambda \times \frac{\delta L(t)}{\delta L(t) + 1}.$$
Note that $\gamma(\cdot, T, T+\delta)$ is an adapted process.

4. Define

$$\begin{aligned}\gamma(t,T+\delta,T) &= \Sigma(t,T)-\Sigma(t,T+\delta)\\ &= \int_T^{T+\delta}\sigma(t,u)\,du.\end{aligned}$$

Choose $\sigma(t,u)$ constant for $u\in[T,T+\delta]$ so that the equation

$$\lambda\times\frac{\delta L(t)}{\delta L(t)+1}=\int_T^{T+\delta}\sigma(t,u)\,du$$

is satisfied; namely, take

$$\begin{aligned}\sigma(t,u) &= \frac{1}{\delta}\lambda\frac{\delta L(t)}{\delta L(t)+1}\\ &= \frac{\lambda L(t)}{\delta L(t)+1}.\end{aligned}$$

5. For $u\notin[T,T+\delta]$ choose $\sigma(t,u)$ arbitrarily. For example, set

$$\sigma(t,u)=\frac{\lambda L(t)}{\delta L(t)+1}$$

for **all** u.

6. Now the model is completely specified under $T+\delta$-forward measure $\mathbf{Q}^{T+\delta}$. Change it to risk-neutral and that is it.

4. Market model of LIBOR rates

In the previous section we constructed an HJM model where a single LIBOR rate followed a lognormal process. It is possible to extend that on a collection of LIBOR rates.

Fix a tenor structure

$$\begin{aligned}T_0 &= 0<T_1<\cdots<T_M,\\ \delta_m &= T_{m+1}-T_m.\end{aligned}$$

Consider a collection of LIBOR rates

$$\{L_m(t)\}_{m=1}^{M-1},$$

where for each m, $L_m(t)$ is a LIBOR rate that resets at T_m and with tenor δ_m (so the corresponding floating cashflows pays at T_{m+1}), so that

$$L_m(t)=\frac{P(t,T_m)-P(t,T_m+\delta)}{\delta_m P(t,T_m+\delta)}=\frac{P(t,T_m)-P(t,T_{m+1})}{\delta_m P(t,T_{m+1})}.$$

Suppose a collection of caplet volatilities $\{\lambda_m\}_{m=1}^{M-1}$ that we want to match is fixed (observed on the market). Then the following theorem holds.

Theorem 4.1 (LIBOR market model)**.** *There exists an HJM model on a probability space* $(\Omega, \mathcal{F})$ *with risk-neutral measure* $\mathbf{Q}$ *such that for every* m*,* $m = 1, \dots, M-1$*,*

$$dL_m(t) = \lambda_m L_m(t)\, dW_t^{T_{m+1}},$$

where $W_t^{T_{m+1}}$ *is a Brownian motion under* T_{m+1}*-forward measure* $\mathbf{Q}^{T_{m+1}}$*. The HJM model (that is, the collection of forward rate volatilities* $\sigma(\cdot, .)$*) is **not** uniquely defined by caplet volatilities* $\{\lambda_m\}_{m=1}^{M-1}$*.*

Proof. Goes pretty much like the one in our simple example for a single LIBOR rate. For details see [MR, Chapter 14]. ∎

Each LIBOR rate follows a lognormal process *under its own measure* $\mathbf{Q}^{T+\delta}$. This guarantees that Black's assumptions are satisfied. However, it makes evaluation of instruments that depend on *more than one* LIBOR rate quite difficult. It would be much more convenient if we knew the simultaneous dynamics of *all* LIBOR rates under a *single* measure. It turns out that risk-neutral measure is not a convenient measure for LIBOR market models, so something else would be useful.

Jamshidian (see [J]) was the first one to construct such a universal measure. He called it a *spot LIBOR measure*.

Recall that risk-neutral measure corresponds to the choice of money-market account as a numeraire. In a money market account, the money is constantly reinvested at short rate.

Spot LIBOR measure corresponds to a "discretely compounded numeraire". The money is reinvested at LIBOR rates at times T_m for the next time period $[T_m, T_{m+1}]$. if we start with \$1 at time T_0, then the value of the discretely-compounded money-market account is given by

$$\begin{aligned}
G_{T_0} &= 1, \\
G_{T_1} &= G_{T_0}\left(1 + \delta_0 L_0(T_0)\right), \\
G_{T_2} &= G_{T_1}\left(1 + \delta_1 L_1(T_1)\right), \\
&\dots .
\end{aligned}$$

Note that

$$1 + \delta_j L_j(T_j) = \frac{1}{P(T_j, T_{j+1})}$$

so that

$$G_{T_m} = \prod_{j=0}^{m-1} \left(1 + \delta_j L_j\left(T_j\right)\right) = \prod_{j=1}^{m} P^{-1}\left(T_{j-1}, T_j\right).$$

In between "rollover" dates $\{T_m\}_{m=0}^{M}$, G_t is uniquely specified by no-arbitrage arguments. If

$$T_{m-1} < t < T_m,$$

then

$$G_t = P\left(t, T_m\right) \cdot G_{T_m}.$$

Define a deterministic function ("index of a first rollover date after t")

$$m\left(t\right) = \inf\left\{k \in \mathbb{Z} : T_k \geq t\right\}.$$

Then

$$G_t = P\left(t, T_{m(t)}\right) \prod_{j=1}^{m(t)} P^{-1}\left(T_{j-1}, T_j\right).$$

Definition 4.1. *A* spot Libor measure $\bar{\mathbf{Q}}^L$ *is a measure that corresponds to G_t being a numeraire, namely the measure under which*

$$\frac{P\left(t, T_m\right)}{G_t}$$

is a martingale for each T_m, $m = 1, \ldots, M$.

Theorem 4.2 (On spot LIBOR measure)**.** *The dynamics of the LIBOR rates under spot Libor measure $\bar{\mathbf{Q}}^L$ are given by*

(4.1)

$$dL_j\left(t\right) = \sum_{k=m(t)}^{j} \frac{\delta_{k+1}\lambda_k\lambda_j L_k\left(t\right) L_j\left(t\right)}{1 + \delta_{k+1} L\left(t, T_k\right)} dt + \lambda_j L_j\left(t\right) d\bar{W}_t^L, \quad j = 1, \ldots, M-1,$$

where $\bar{W}_t^L$ is a Brownian motion under $\bar{\mathbf{Q}}^L$.

LIBOR rates are of course no longer martingales under $\bar{\mathbf{Q}}^L$. However, the drifts in (4.1) are still expressed in terms of "observables". An important practical conclusion is that once we specify caplet volatilities $\{\lambda_m\}_{m=1}^{M-1}$, we do not have to backup instantaneous forward volatilities $\sigma\left(\cdot, \cdot\right)$ from them; we can use $\{\lambda_m\}_{m=1}^{M-1}$ and (4.1) to completely and consistently specify the evolution of LIBOR rates $\{L_m\left(t\right)\}_{m=1}^{M-1}$ under the same measure.

Also note that the numeraire G_t is also specified in terms of observables, so we can "evolve forward" LIBOR rates and the numeraire simultaneously (say in Monte-Carlo), only knowing caplet volatilities.

5. Market model of swap rates

In [L2] we constructed a swap measure, i.e. a measure under which swaptions are valued by Black-like expression. For a swaption we have

$$V_t = B_t \mathbf{E}\left(B_{t_1}^{-1} \left((1 - P(t_1, t_K)) - C \sum_{i=2}^{K} P(t_1, t_i)\, \tau_i \right)^+ \middle| \mathcal{F}_t \right).$$

Using the value of the fixed leg as a numeraire,

$$N_t = \sum_{i=2}^{K} P(t, t_i)\, \tau_i,$$

we get

$$\begin{aligned} V_t &= N_t \hat{\mathbf{E}}^N \left(N_{t_1}^{-1} \left((1 - P(t_1, t_K)) - C \sum_{i=2}^{K} P(t_1, t_i)\, \tau_i \right)^+ \middle| \mathcal{F}_t \right) \\ &= \left[\sum_{i=2}^{K} P(t, t_i)\, \tau_i \right] \hat{\mathbf{E}}^N \left((F_{t_1} - C)^+ \middle| \mathcal{F}_t \right), \end{aligned}$$

where F_{t_1} is the swap rate,

$$F_t = \frac{P(t, t_1) - P(t, t_K)}{\sum_{i=2}^{K} P(t, t_i)\, \tau_i}.$$

The swap rate is a martingale under the swap measure (why?). It is pretty clear that we can find HJM model under which the swap rate is actually log-normal. The procedure is quite similar to what we did for LIBOR.

In the same way, we can construct a model in which a collection of swap rates all have log-normal distributions under the appropriate measures. As the formulas are unwieldy we refer the interested reader to [MR] and [J].

Future discussion will concern market models of LIBOR rates, but all considerations apply to market models of swap rates as well.

6. Market model of LIBOR and swap rates

Does not exist.

7. Calibration issues

We are going to just barely scratch the surface of a very complex issue of calibrating market models. Reading Rebonato's books [R1] and [R2] is mandatory for anybody interested in the subject. Also, Appendix gives an illuminating example.

In market models, volatilities of LIBOR rates need not be constant, nor the number of factors need be 1. In the most general form we can write the following joint system for stochastic evolution of LIBOR rates $\{L_j(t)\}_{j=1}^{M-1}$,

(7.1)

$$dL_j(t)/L_j(t) = \mu_j(t)\,dt + \sum_{n=1}^{N} a_{jn}(t)\,dZ_n(t), \quad j = 1, \ldots, M-1.$$

It does not really matter what measure we use; this general form will hold under any measure. The difference between different measures will be in different drifts $\{\mu_j(t)\}_{j=1}^{M-1}$, that are in general will not be deterministic functions, but some rather complex expressions involving rates, volatilities, correlations, etc.

In the formula (7.1), N is the number of factors, $\{Z_n\}_{n=1}^{N}$ are independent Brownian motions (under whatever measure we are working in), $a_{jn}(t)$ are deterministic, time-dependent, instantaneous volatilities of LIBOR rates.

Calibration is a process of specifying the time-dependent matrix of instantaneous volatilities $\{a_{jn}(t); \quad j = 1, \ldots, M-1, \quad n = 1, \ldots, N, \quad t \geq 0\}$ so that market prices of some instruments are recovered by the model.

First and most important step in calibration is to make sure that the prices of caplets that correspond to the rates $\{L_j(t)\}_{j=1}^{M-1}$ are recovered (after all, this is why we have market models). This is relatively easy. To match the price of a caplet we just need to make sure that the total volatility of a specific LIBOR rate, as given by the model, matches the caplet Black's volatility. The following constraints have to be satisfied:

$$\mathrm{Var}\,[\log L_j(T_j)] = \lambda_j^2, \quad j = 1, \ldots, M-1.$$

From (7.1), the variances are easily calculated,

$$\sum_{n=1}^{N} \int_0^{T_j} a_{jn}^2(t)\,dt = \lambda_j^2, \quad j = 1, \ldots, M-1.$$

There seems to be an awfully small number of constraints for a time-dependent matrix of coefficients. Unfortunately, this is about all we can do with market-implied information. The universe of actively traded

LIBOR-rate dependent instruments is very, very small, and there are no other high-volume instruments to provide calibration data.

One can try to add information from the only other high-volume sector, swaptions. This generally creates more problems than it solves[1].

Another possible set of constraints may come from historical estimation. One can try to match historically observed correlation with (7.1). Correlation among LIBOR rates is easy to compute in the model. For instantaneous correlations we have

$$\text{Corr}\left[\frac{dL_j(t)}{L_j(t)}, \frac{dL_m(t)}{L_m(t)}\right] = \sum_{n=1}^{N} a_{jn}(t)\, a_{jm}(t).$$

To get term correlation to date T, we integrate this formula,

$$\text{Corr}\left[\log L_j(T), \log L_m(T)\right] = \sum_{n=1}^{N} \int_0^T a_{jn}(t)\, a_{jm}(t)\; dt.$$

Of course, a number of problems arise immediately. Among them, what is the time horizon T to use? How to handle very unstable historical correlations? and so on.

In the face of a problem of having too many parameters to calibrate, and too few constraints to calibrate to, the most natural reaction is to begin imposing external constraints. We can take instantaneous volatilities constant, we can take fewer factors. It is quite clear that the potential for overfitting is enormous.

- Market models do not provide any structure to the dynamics of rates;
- They do not provide any connection between various "pieces of the puzzle";
- Having so many parameters, they have no predictive power;
- They have no potential to show any mispricing in the market because they happily match whatever prices are thrown at them.

More on this subject can (and should!) be read in [R1, Chapter 18.5].

It does not of course mean that market models are useless. What it means is that they have to be coupled with something else, something that provides a sensible and stable description of the term structure of

[1]Academic journals are awash in papers trying to calibrate models to both caps/floors AND swaptions. The reality of the matter is, however, that these two markets exhibit very little connection. A number of high-profile disasters involving attempts to cross-trade caps/swaptions have recently been reported. As there are no trading connection between the two markets, there is no modelling connection either.

caplet volatilities/correlations. This something has to come from an *external* source. Market models themselves cannot provide it because of their extreme flexibility. As a plausible approach, PCA as applied by Yuri to the statistical model can be married to market models.

8. Appendix

Let us present a simple yet illuminating example on calibrating a market model. It is lifted, pretty much unchanged, from [R2].

We will disregard what was said earlier, that theoretically a model where LIBOR rates and swap rates both have log-normal distribution does not exist. Consider a tenor structure

$$\begin{aligned} 0 &= T_0 < T_1 < T_2 < T_3, \\ \delta_m &= T_{m+1} - T_m. \end{aligned}$$

A LIBOR rate L_1 resets at date T_1 and pays at T_2. A LIBOR rate L_2 resets at T_2 and pays at T_3. In addition to these two rates, we have a swap rate S, for a swap that begins at T_1 and pays at T_2 and T_3. In terms of the relevant bonds, we have

$$\begin{aligned} L_1(t) &= \frac{P(t,T_1) - P(t,T_2)}{\delta_1 P(t,T_2)}, \\ L_2(t) &= \frac{P(t,T_2) - P(t,T_3)}{\delta_2 P(t,T_3)}, \\ S(t) &= \frac{P(t,T_1) - P(t,T_3)}{\delta_1 P(t,T_2) + \delta_2 P(t,T_3)}. \end{aligned}$$

We can observe three volatilities from the market. One is the volatility of (the log of) $L_1(t)$ from 0 to T_1. Call it σ_1. The other is the volatility of (the log of) $L_1(t)$ from 0 to T_2, call it σ_2. Finally, from the swaptions's market, we get the volatility of (the log of) $S(t)$ from 0 to T_1, call it ν.

We would like to use this data to "calibrate" a market model for L_1 and L_2. We can assume that in between dates T_m, their instantaneous volatilities are flat. Therefore, to build a model, we have to specify (see Figure 1)

- How the log-normal volatility of L_2 is "split" between $[0,T_1]$ and $[T_1,T_2]$;
- What is the correlation (call it ρ) of L_1 and L_2 over the interval $[0,T_1]$.

How do we relate the swap rate to LIBOR rates? Easy,

$$\begin{aligned}
S(t) &= \frac{P(t,T_1)-P(t,T_3)}{\delta_1 P(t,T_2)+\delta_2 P(t,T_3)} \\
&= \frac{P(t,T_1)-P(t,T_2)+P(t,T_2)+P(t,T_3)}{\delta_1 P(t,T_2)+\delta_2 P(t,T_3)} \\
&= \frac{\delta_1 P(t,T_2)}{\delta_1 P(t,T_2)+\delta_2 P(t,T_3)} \times \frac{P(t,T_1)-P(t,T_2)}{\delta_1 P(t,T_2)} \\
&\quad + \frac{\delta_2 P(t,T_3)}{\delta_1 P(t,T_2)+\delta_2 P(t,T_3)} \times \frac{P(t,T_2)-P(t,T_3)}{\delta_2 P(t,T_3)} \\
&= w_1 L_1(t) + w_2 L_2(t).
\end{aligned}$$

We can assume that w_1 and w_2 ($w_{1,2} > 0$, $w_1 + w_2 = 1$) are constant over time (tests show that they are much less volatile than the rates, so the approximation is reasonable). For $t = T_1$ we then have,

$$S(T_1) = w_1 L_1(T_1) + w_2 L_2(T_1).$$

Taking variance of both sides we get,

$$\operatorname{Var} S(T_1) = w_1^2 \operatorname{Var} L_1(T_1) + w_2^2 \operatorname{Var} L_2(T_1) + 2w_1w_2 \operatorname{Covar} L_1(T_1) L_2(T_1).$$

Then (approximately)

$$S^2\nu^2 T_1 = w_1^2 L_1^2 \sigma_1^2 T_1 + w_2^2 L_2^2 (\sigma_2')^2 T_1 + 2w_1w_2 L_1 L_2 \sigma_1 \sigma_2' \rho T_1.$$

Here, everything can be market implied except for ρ (correlation between LIBOR rates over $[0,T_1]$) and log-normal volatility σ_2' of L_2 over $[0,T_1]$. This is a very important point to understand: *this volatility is not available from the market*!

Another equation connects volatility of L_2 over $[0,T_1]$ and $[T_1,T_2]$ (call the latter σ_2''). We have thus two equations,

$$\begin{aligned}
S^2\nu^2 &= w_1^2 L_1^2 \sigma_1^2 + w_2^2 L_2^2 (\sigma_2')^2 + 2w_1w_2 L_1 L_2 \sigma_1 \sigma_2' \rho, \\
\sigma_2^2 T_2 &= (\sigma_2')^2 T_1 + (\sigma_2'')^2 (T_2 - T_1).
\end{aligned}$$

Two equations and three unknowns (ρ, σ_2', σ_2''). Thus, we have an infinum combinations to satisfy the equations,

- Do we take $\rho = 1$ and choose $\sigma_2'' \neq \sigma_2'$; or
- Do we take $\sigma_2'' = \sigma_2'$ and ρ strictly less than 1; or
- Something in between?

There is virtually no other market information we can use, so it is pretty much a judgement call.

Why the choice is important, however? Consider a periodic (rachet) caplet, an instrument that pays

$$\max\left\{L_2\left(T_2\right)-L_1\left(T_1\right),0\right\}$$

at time T_3. Market in such instruments is very thin, but clients do ask their brokers to quote prices. It is quite clear that a periodic caplet will derive most of its value from volatility σ_2'', so making a right choice (between perfect correlation and constant vol) is very important.

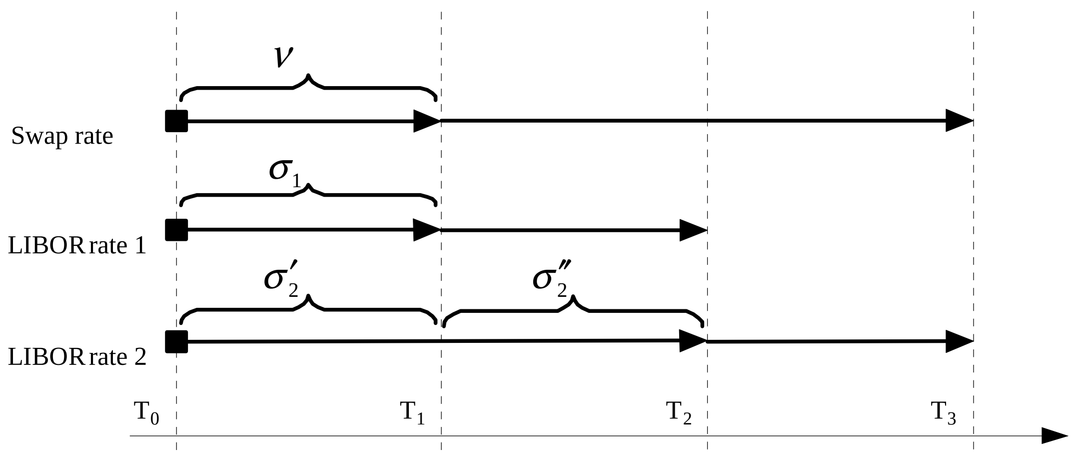

Figure 1

Hopefully, this toy example demonstrates the mind-boggling complexity of calibrating a market model.

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