FORWARD MEASURES AND CHANGES OF NUMERAIRE

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1. Introduction

In the second lecture of the series (see [L1]) we go deeper into the realm of interest-rate models. Our goal is simple: derive *useful* valuation formulas for derivative securities.

2. Setup

Recall the notations of the previous lecture. We assume that a probability space $(\Omega, \mathcal{F})$ is given, equipped with a *risk-neutral* measure $\mathbf{Q}$. The real-world measure will never be mentioned in this lecture.

We also fix a Brownian motion W_t (note that we remove "tilde" from the notation of previous lecture for clarity), and assume that the filtration $\{\mathcal{F}_t, \quad 0 \leq t < \infty\}$ is Brownian. We present all the results in one-factor framework, which is only done for brevity. We denote bonds by $P(t,T)$ and assume them to follow the equations

$$dP(t,T) = r(t) P(t,T)\, dt + P(t,T) \Sigma(t,T)\, dW_t$$

with the appropriate technical conditions imposed on bond volatility processes $\Sigma(t,T)$. We denote by B_t a money-market account; it follows the law

$$dB_t = r(t) B_t\, dt.$$

A security that pays X at time T has value at time t that is equal to

$$\pi_t(X) = B_t \mathbf{E}\left(B_S^{-1} X \middle| \mathcal{F}_t \right). \tag{2.1}$$

A number of different measures will be introduced in the lecture. Each one will be denoted by letter $\mathbf{Q}$ with some sort of decoration (for example $\hat{\mathbf{Q}}$). The expected value operator that corresponds to a measure like that will be denoted by letter $\mathbf{E}$ with the same decoration (for example for $\hat{\mathbf{Q}}$ it would be $\hat{\mathbf{E}}$).

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3. Motivational example

Probably the simplest possible derivative security we can think of is an option on a discount bond (remember that a caplet is actually an option on a discount bond, so this is a real-world example). Suppose we have a call option that expiries at time T on a bond $P(\cdot, M)$, where $0 < T < M$. Suppose the strike is K. The payoff of the option is

$$C = (P(T, M) - K)^{+},$$

paid at time T. Clearly C is $\mathcal{F}_T$-measurable (because C depends only on the value of the bond at time T). Then by (2.1),

$$\begin{aligned} \pi_{\mathbf{0}}(C) &= \mathbf{E}\left(B_T^{-1} C\right) \\ &= \mathbf{E}\left(B_T^{-1}(P(T, M) - K)^{+}\right). \end{aligned} \tag{3.1}$$

If we actually needed to compute this value, we would have to know the joint distribution (at time $t = 0$) of *two* random variables, B_T^{-1} and $P(T, M)$. We would then have to do *two*-dimensional integration of the payoff with respect to their joint density function. Seems like a lot of trouble for a simple option on a discount bond. Can we simplify this formula somehow? Recall that in Black's model we would just have the expression (for example, assuming log-normal distribution for the bond)

$$\mathrm{Black}_{\mathbf{0}}(C) = P(0, T)\, \hat{\mathbf{E}}\,(P(T, M) - K)^{+} \tag{3.2}$$

(we were careful to use a different letter $\hat{\mathbf{E}}$ to denote different assumptions of Black's model). The difference between models (3.1) and (3.2) is that in the latter, we only have to specify a *one*-dimensional distribution (of $P(T, M)$) and perform a *one*-dimensional integration with respect to bond's density. Further comparison of the two formulas reveals that in Black's formula we are able to "pull" B_T^{-1} from under the expectation sign and replace it with a (non-random) quantity $P(0, T)$.

Can this action be somehow justified?

4. Measure change

4.1. **1D example.** We have performed a change of measure when we switched from real-world measure $\mathbf{P}$ (oops! I said earlier that real world measure will never be mentioned in this lecture.. But this is the only time, I promise) to a risk-neutral one $\mathbf{Q}$ in the previous lecture. The utility of that little trick has not been completely exhausted, however. Let us look at a simple measure change example.

Let X be a random variable with density $f(x)$. Let $\phi(x)$ be a real-valued function. What is the expected value of the random variable

$\phi(X)$? Well, this is easy,

$$\mathbf{E}_f \phi(X) = \int_{-\infty}^{\infty} \phi(x) f(x)\ dx.$$

Let $g(x)$ be another function such that it is always positive and integrates to 1.0, so that

$$g(x) > 0 \text{ for all } x,$$
$$\int_{-\infty}^{\infty} g(x)\ dx = 1.$$

Then we can do the following stupid little transformation,

$$\begin{aligned} \mathbf{E}_f \phi(X) &= \int_{-\infty}^{\infty} \phi(x) f(x)\ dx \\ &= \int_{-\infty}^{\infty} \phi(x) f(x) (g(x)/g(x))\ dx \\ &= \int_{-\infty}^{\infty} \phi(x) \frac{f(x)}{g(x)} g(x)\ dx. \end{aligned} \tag{4.1}$$

Define

$$\psi(x) = \phi(x) \frac{f(x)}{g(x)}.$$

Then we see that

$$\mathbf{E}_f \phi(X) = \int_{-\infty}^{\infty} \psi(x) g(x)\ dx$$

which looks like the expected value of $\psi(X)$ under the measure in which random variable X has density $g(\cdot)$,

$$\begin{aligned} \mathbf{E}_f \phi(X) &= \mathbf{E}_g \psi(X) \\ &= \mathbf{E}_g \left[\phi(X) \frac{f(X)}{g(X)} \right]. \end{aligned} \tag{4.2}$$

The term $\frac{f(x)}{g(x)}$ represents a density of measure $\mathbf{E}_f$ with respect to measure $\mathbf{E}_g$, often called a Radon-Nikodim derivative.

It is completely not clear that simple rearranging of terms as in (4.1), (4.2) has any value whatsoever. It probably does not in the example above. But the example demonstrates a very deep point. If we read equation (4.2) from "right" to "left", then we see that we can simplify the expression under the expectation sign by an appropriate measure change.

4.2. **Static theory.** Given measure $\mathbf{Q}$ on the probability space $(\Omega, \mathcal{F})$ and a random variable Z such that

$$\begin{aligned} Z > 0 \quad & \mathbf{Q}\text{-a.s.}, \\ \mathbf{E}Z = 1, & \end{aligned}$$

we can define another measure $\bar{\mathbf{Q}}$ on the same probability space by

$$\begin{aligned} \bar{\mathbf{Q}}(A) &\triangleq \mathbf{E}\left(Z \times 1_{\{A\}}\right) \\ &= \int_A Z(\omega)\ \mathbf{Q}(d\omega) \end{aligned} \tag{4.3}$$

(here $A \in \mathcal{F}$ and $1_{\{A\}}$ is an indicator of an event A). First question is of course whether $\bar{\mathbf{Q}}$ is a probability measure at all.

HW Problem 1: Prove that it is! At least check the following conditions (here $\emptyset$ is the null event)

- $\bar{\mathbf{Q}}(\emptyset) = 0$, $\bar{\mathbf{Q}}(\Omega) = 1$;
- For any $A \in \mathcal{F}$, $\bar{\mathbf{Q}}(A) \geq 0$;
- If $\{A \text{ and } B\} = \emptyset$ then $\bar{\mathbf{Q}}(A \text{ or } B) = \bar{\mathbf{Q}}(A) + \bar{\mathbf{Q}}(B)$, for events A, $B \in \mathcal{F}$.

Random variable Z plays the role of a density (Radon-Nikodim derivative) of measure $\bar{\mathbf{Q}}$ with respect to measure $\mathbf{Q}$. This fact is usually formally described as

$$\frac{d\bar{\mathbf{Q}}}{d\mathbf{Q}} = Z. \tag{4.4}$$

The relation (4.3) can also be written for expected values as

$$\begin{aligned} \bar{\mathbf{E}}\xi &= \mathbf{E}(\xi Z) \\ &= \int_\Omega \xi(\omega) \frac{d\bar{\mathbf{Q}}}{d\mathbf{Q}}(\omega)\ \mathbf{Q}(d\omega), \end{aligned} \tag{4.5}$$

here ξ is a random variable.

4.3. **Dynamic theory.** Conditional expectations play a very important role in stochastic calculus in general and financial math in particular. We would like the change of measure to "respect" conditional expectations. We would like to make sure that something akin to (4.5) holds not only for unconditional but for *conditional* expected values as well.

For general Z, measure change as defined in the previous section cannot be extended on conditional expectations. To remedy that we must further restrict the class of measure-change densities allowed.

Let $\{Z_t\}_{t=0}^{\infty}$ be a positive martingale with initial value 1, so that

$$Z_t > 0 \quad \mathbf{Q}\text{-a.s.},$$
$$\mathbf{E}\left(Z_t \middle| \mathcal{F}_s\right) = Z_s, \quad t \geq s,$$
$$Z_0 = 1.$$

Then for any T, $T > 0$,

$$\mathbf{E} Z_T = 1.$$

We therefore can define a measure $\bar{\mathbf{Q}}_T$ on sigma-algebra $\mathcal{F}_T$ by the standard formula

$$\bar{\mathbf{Q}}_T(A) \triangleq \mathbf{E}\left(Z_T \times 1_{\{A\}}\right), \quad A \in \mathcal{F}_T. \tag{4.6}$$

This is conveniently written as (compare to (4.4))

$$\left.\frac{d\bar{\mathbf{Q}}}{d\mathbf{Q}}\right|_{\mathcal{F}_T} = Z_T. \tag{4.7}$$

HW Problem 2: Recall that $\mathcal{F}_t \subset \mathcal{F}_T$ for $t \leq T$. If $A \in \mathcal{F}_t$ then $A \in \mathcal{F}_T$ as well, and we have two probabilities defined for A by using (4.6),

$$\bar{\mathbf{Q}}_t(A) = \mathbf{E}\left(Z_t \times 1_{\{A\}}\right)$$

and

$$\bar{\mathbf{Q}}_T(A) = \mathbf{E}\left(Z_T \times 1_{\{A\}}\right)$$

We must, of course, have that (*consistency* condition)

$$\bar{\mathbf{Q}}_t(A) = \bar{\mathbf{Q}}_T(A) \quad \text{for } A \in \mathcal{F}_t, \quad 0 \leq t \leq T.$$

Prove it.

Having defined measure $\bar{\mathbf{Q}}$ by (4.6), we need to obtain the change-of-measure (Bayes) formula for conditional expectations, which now has the following form:

Theorem 4.1 (Bayes' Rule)**.** *If $0 \leq t \leq T$ and ξ is an $\mathcal{F}_T$-measurable random variable satisfying $\bar{\mathbf{E}}_T|\xi| < \infty$, then*

$$\bar{\mathbf{E}}\left(\xi \middle| \mathcal{F}_t\right) = \frac{1}{Z_t}\mathbf{E}\left(Z_T\xi \middle| \mathcal{F}_t\right). \tag{4.8}$$

HW Problem 3: Prove the theorem. Hint: Recall the definition of a conditional expected value $\bar{\mathbf{E}}\left(\xi \middle| \mathcal{F}_t\right)$ and show that the expression on the right-hand side of (4.8) satisfies it.

5. Forward measure

5.1. **Fishing expedition.** Having forayed this far into heavy theory, what use can we make of it? Recall our basic valuation equation

$$\pi_t(X) = B_t \mathbf{E}\left(B_T^{-1} X \middle| \mathcal{F}_t \right) \tag{5.1}$$

for $\mathcal{F}_T$-measurable payoffs X and our desire to "take out" B_T^{-1} from under the expected value operator.

Comparing the right-hand-sides of (5.1) and (4.8) we see undeniable similarities. We can identify X with ξ (both are $\mathcal{F}_T$-measurable). We would like to identify B_T^{-1} with Z_T but here is a snag: $\left\{B_t^{-1}\right\}$ is NOT a martingale under $\mathbf{Q}$. Measure changes only work with martingales.

There is a neat trick that allows us to work around that. Recall from [L1] that the discounted bond value

$$Z(t,T) = \frac{P(t,T)}{B_t}$$

is a $\mathbf{Q}$-martingale. Then

$$\begin{aligned} B_t &= \frac{P(t,T)}{Z(t,T)}, \\ B_T &= \frac{P(T,T)}{Z(T,T)} \\ &= \frac{1}{Z(T,T)}, \\ B_T^{-1} &= Z(T,T). \end{aligned}$$

We can rewrite (5.1) as

$$\pi_t(X) = \frac{P(t,T)}{Z(t,T)} \mathbf{E}\left(Z(T,T) \cdot X \middle| \mathcal{F}_t \right).$$

Note that the term

$$\frac{\mathbf{E}\left(Z(T,T) \cdot X \middle| \mathcal{F}_t \right)}{Z(t,T)}$$

is exactly the same as the right-hand side of (4.8) if we identify $Z(t,T) \sim Z_t$, $X \sim \xi$. Note that $Z(t,T) > 0$ as required by the measure change theory.

There is very minor problem still, since in general $Z(0,T)$ is not equal to 1. We can fix that my simple scaling.

5.2. **Definition and first reaction.** Previous section gave us a motivation for doing the measure change. Here we give a formal definition. Measure $\mathbf{Q}^T$ (with expected value operator $\mathbf{E}^T$) is called T-forward measure if it is defined by

$$\left.\frac{d\mathbf{Q}^T}{d\mathbf{Q}}\right|_{\mathcal{F}_t} = \frac{Z(t,T)}{Z(0,T)},$$

where $Z(t,T)$ is the discounted bond value process and $\mathbf{Q}$ is a risk-neutral measure.

The definition is valid since the process

$$\left\{\frac{Z(t,T)}{Z(0,T)}\right\}_{t=0}^{T}$$

is a positive normalized martingale.

Theorem 5.1 (Forward Measure Pricing Formula)**.** *If X is a payoff that is $\mathcal{F}_T$-measurable, then*

$$\pi_t(X) = P(t,T)\,\mathbf{E}^T(X|\,\mathcal{F}_t),$$

where $\mathbf{Q}^T$ is T-forward measure as defined above, and $\mathbf{E}^T$ is the expected value operator with respect to that measure.

HW Problem 4: Prove the theorem. Hint: Apply Bayes' Rule.

The theorem shows that we can achieve our main goal of "decoupling" the money market B_T and payoff X in (2.1), at the expense of using a measure different from risk-neutral. This is usually a small price to pay.

It is clear from the definition that measure $\mathbf{Q}^T$ depends on a particular time T. So in effect we have a whole collection of forward measures $\left\{\mathbf{Q}^T\right\}_{T=0}^{\infty}$ indexed by time T. *Successful application of forward measure critically depends on the identification of the right time T.* Usually T is taken to be the expiry time of an option, so that the payoff is measurable with respect to $\mathcal{F}_T$.

5.3. **Example revisited.** With the help of forward measure, we can rewrite (3.1) as

$$\begin{aligned} \pi_0(C) &= \mathbf{E}\left(B_T^{-1}\left(P(T,M)-K\right)^+\right) \\ &= P(0,T)\,\mathbf{E}^T\left(\left(P(T,M)-K\right)^+\right) \end{aligned} \tag{5.2}$$

which is exactly Black's formula (3.2). Black's model is not that stupid after all!

We will show later in this course that Black's formula, as applied to caps, swaptions and other instruments, can in fact be rigorously justified in HJM framework *by the appropriate change of measure.* These results provide a theoretical foundation for now-fashionable "market models" (of which BGM is one example).

6. Properties of forward measure

Lemma 6.1. *Let $F(t,T,M)$ be the forward price of M-maturity bond with settlement at time T, so that*

$$F(t,T,M) = \frac{P(t,M)}{P(t,T)}.$$

Then

$$F(t,T,M) = \mathbf{E}^T\left(P(T,M)|\,\mathcal{F}_t\right).$$

Proof. (Mirrors closely the solution of the first problem in the last homework). By definition of a forward price, $F(t,T,M)$ is an $\mathcal{F}_t$-measurable random variable such that the value of a contingent claim with payoff (at time T)

$$X = P(T,M) - F(t,T,M)$$

at time t is 0,

$$0 = \pi_t\left(P(T,M) - F(t,T,M)\right).$$

By Forward Measure Pricing Formula

$$\pi_t\left(P(T,M) - F(t,T,M)\right) = P(t,T)\,\mathbf{E}^T\left(X|\,\mathcal{F}_t\right),$$

so that

$$0 = P(t,T)\,\mathbf{E}^T\left(P(T,M) - F(t,T,M)|\,\mathcal{F}_t\right).$$

The random variable $F(t,T,M)$ is $\mathcal{F}_t$-measurable so

$$\mathbf{E}^T\left(F(t,T,M)|\,\mathcal{F}_t\right) = F(t,T,M).$$

Therefore,

$$0 = \mathbf{E}^T\left(P(T,M)|\,\mathcal{F}_t\right) - F(t,T,M),$$

and

$$F(t,T,M) = \mathbf{E}^T\left(P(T,M)|\,\mathcal{F}_t\right).$$

∎

Lemma 6.2. *The process*

$$\{F(t,T,M), \quad t \in [0,T]\}$$

is a martingale under T-forward measure $\mathbf{Q}^T$.

Proof. Immediately follows from the previous lemma and the tower rule for conditional expected values. ∎

Lemma 6.3. *Let A_t be the price at time t of a traded asset, so that*

$$\left\{B_t^{-1}A_t, \quad t \geq 0\right\}$$

is a martingale under risk-neutral measure $\mathbf{Q}$. Let $F_A(t,T)$ be its forward price at t for settlement at T. We know from homework that

$$F_A(t,T) = \frac{A_t}{P(t,T)}.$$

Then

- *the forward price can also be computed as*
$$F_A(t,T) = \mathbf{E}^T\left(A_T \middle| \mathcal{F}_t\right);$$
- *the process of forward price of A,*
$$\{F_A(t,T), \quad t \geq 0\}$$
is a martingale under T-forward measure $\mathbf{Q}^T$.

HW Problem 5: Prove the lemma.

These results explain why $\mathbf{Q}^T$ is called "T-forward measure". It is identified by the fact that forward prices (settled at time T) of all traded instruments are martingales with respect to it. In fact, this property identifies the measure uniquely.

7. Forward measure in Gaussian HJM

7.1. **Gaussian HJM.** A Gaussian HJM model is an HJM model (see [L1]) where forward rate volatilities $\sigma(t,T)$ are *deterministic functions.* For example, Ho-Lee, Hull-White and Yuri's model (after some modifications) are Gaussian HJM's. This is a widely used subclass of HJM models, appreciated mostly for extensive analytical tractability, a consequence of the fact that all forward rates are distributed normally, and all bonds are distributed log-normally (check the formulas in [L1] to convince yourself that this is indeed the case).

Forward measures are especially easy to characterize in Gaussian HJM models. We will deal with one-factor Gaussian HJM models for brevity, but the procedure is extendable. We shall start with the following definitions (under risk-neutral measure $\mathbf{Q}$),

$$\begin{aligned} df(t,T) &= -\Sigma(t,T)\,\sigma(t,T)\,dt + \sigma(t,T)\,dW_t, \\ dP(t,T) &= r(t)\,P(t,T)\,dt + P(t,T)\,\Sigma(t,T)\,dW_t, \end{aligned}$$

where $\sigma(t,T), \Sigma(t,T)$ are deterministic functions of their two variables (and of course

$$\sigma(s,T) = -\frac{\partial}{\partial T}\Sigma(s,T)$$

as in the first lecture).

7.2. **Forward measure derivation.** We have mentioned in the previous sections that T-forward measure is uniquely defined by the fact that all T-forward prices are martingales under it. Consider a bond that expiries at time M. Then its evolution is governed by the equation

$$dP(t,M) = r(t)\,P(t,M)\;dt + P(t,M)\,\Sigma(t,M)\;dW_t.$$

From last homework we know that

$$\begin{aligned} dF(t,T,M) &= F(t,T,M)\left[\Sigma^2(t,T) - \Sigma(t,T)\,\Sigma(t,M)\right]\,dt \\ &\quad + F(t,T,M)\left[\Sigma(t,M) - \Sigma(t,T)\right]\,dW_t. \end{aligned} \tag{7.1}$$

This equation holds under risk-neutral measure $\mathbf{Q}$. We would like to change the measure so that $F(t,T,M)$ becomes a martingale. By Girsanov's theorem, measure changes in our setting are equivalent to adding (or subtracting) a drift from the Brownian motion. We want to change the drift of W in (7.1) so that the "dt" part of the equation (7.1) for $F(t,T,M)$ disappears.

The drift change must not of course depend on M because the same drift change should work for all forward processes. It can, however, depend on T. We can rearrange (7.1) so that

$$dF(t,T,M) = F(t,T,M)\left[\Sigma(t,M) - \Sigma(t,T)\right]\left(-\Sigma(t,T)\;dt + dW_t\right).$$

This immediately yields the required measure change. We have proven the following theorem.

Theorem 7.1 (Forward measure in Gaussian HJM)**.** *We have that*

- *The T-forward measure $\mathbf{Q}^T$ is (uniquely) identified by the condition that*

$$dW_t^T \triangleq -\Sigma(t,T)\;dt + dW_t$$

is a (driftless) Brownian motion under $\mathbf{Q}^T$.
- *The T-forward bond price evolution is given by*

$$dF(t,T,M) = F(t,T,M)\left[\Sigma(t,M) - \Sigma(t,T)\right]\,dW_t^T$$

under $\mathbf{Q}^T$. In particular, it is a positive martingale.

- *Under $\mathbf{Q}^T$ the forward bond price $F(t,T,M)$ has a log-normal distribution, so that*
$$\log \frac{F(t,T,M)}{F(0,T,M)} \text{ is Gaussian;}$$
$$\mathbf{E}^T \log \frac{F(t,T,M)}{F(0,T,M)} = -\frac{1}{2}\mathrm{Var}\ \log \frac{F(t,T,M)}{F(0,T,M)},$$
$$\mathrm{Var}\ \log \frac{F(t,T,M)}{F(0,T,M)} = \int_0^t (\Sigma(s,M) - \Sigma(s,T))^2\, ds.$$

Evolution of other quantities of interest under $\mathbf{Q}^T$ can easily be deduced from their evolutions under $\mathbf{Q}$ by replacing dW with $dW^T + \Sigma(\cdot,T)$. Here is an example.

Corollary 7.2. *A bond with maturity S follows the equation*

$$dP(t,S) = (r(t) + \Sigma(t,T)\Sigma(t,S))\, P(t,S)\, dt + P(t,S)\,\Sigma(t,S)\, dW_t^T,$$

and an instantaneous forward rate with maturity S follows the equation
$$df(t,S) = (\Sigma(t,T) - \Sigma(t,S))\,\sigma(t,S)\, dt + \sigma(t,S)\, dW_t^T,$$
under T-forward measure $\mathbf{Q}^T$, where as before
$$\sigma(t,T) = -\frac{\partial}{\partial T}\Sigma(t,T).$$

7.3. **The Hull-White model.** The Hull-White (and its particular case, Ho-Lee) model is a Gaussian HJM model. Therefore, these formulas should apply to it. For the Hull-White model

7.3.1. *Zero-coupon bond option.* A zero-coupon bond option with expiry T and strike K on a bond maturing at M, $M > T$, is an instrument with a payoff at T given by
$$(P(T,M) - K)^+ .$$
Its price at time 0 is given by (recall (5.2))
$$\begin{aligned} \pi_0(C) &= P(0,T)\,\mathbf{E}^T\left((P(T,M) - K)^+\right) \\ &= P(0,T)\,\mathbf{E}^T\left((F(T,T,M) - K)^+\right). \end{aligned}$$
The process $\{F(t,T,M), \quad 0 \le t \le T\}$ is a martingale. Moreover (recall the lecture on the Hull-White model) $F(T,T,M)$ is lognormal with the variance
$$\mathrm{Var} \log F(T,T,M) = \sigma^2 b^2 (M-T)\, \frac{1 - e^{-2aT}}{2a}.$$

Note that the volatility is the same irrespective of what measure we compute it under, either the risk neutral or forward (change of measure only affects the drift). Denote

$$\sigma_P = (\operatorname{Var} \log F(T,T,M))^{1/2} = \sigma b(M-T) \left(\frac{1-e^{-2aT}}{2a} \right)^{1/2}.$$

Then the combination of the martingale property and lognormality allows us to write

$$F(T,T,M) = F(0,T,M) \exp\left(\sigma_P X - \sigma_P^2/2\right),$$

where X is a standard Gaussian randopm variable (mean zero, standard deviation 1). So the value of the bond option

$$\pi_0(C) = P(0,T) \int_{-\infty}^{\infty} \left(F(0,T,M) e^{\sigma_P z - \sigma_P^2/2} - K \right)^+ \frac{1}{2\pi} e^{-z^2/2} dz.$$

Can be computed just like in the Black's model,

$$\begin{aligned} \pi_0(C) &= P(0,T)\left(F(0,T,M)N(h) - KN(h-\sigma_P)\right), \\ h &= \frac{1}{\sigma_P} \log \frac{F(0,T,M)}{K} + \frac{\sigma_P}{2}. \end{aligned}$$

7.3.2. *Short-rate state under forward measure.*

HW Problem 6: Prove that under the T-forward measure

$$x(t) = \frac{\sigma^2}{2} \left(b^2(0,t) - e^{a(T-t)} b^2(0,T) + e^{a(T-t)} b^2(t,T) \right) + \sigma \int_0^t e^{-a(t-s)} \, dW_s^T.$$

(Hint: Use the formula for $x(t)$ under the risk-neutral measure from the Hull-White lecture, and then replace dW with dW^T).

Note that (substitute $t=T$)

$$\begin{aligned} x(T) &= \sigma \int_0^T e^{-a(T-s)} \, dW_s^T, \\ x(T) &= x(t) + \frac{\sigma^2}{2} \left(b^2(0,T) - e^{-a(T-t)} b^2(0,t) - b^2(t,T) \right) + \sigma \int_t^T e^{-a(T-s)} \, dW_s^T. \end{aligned}$$

8. Numeraires

8.1. **A general numeraire.** The risk-neutral measure is characterized by the fact that the price of any traded asset divided by money-market account is a martingale under it. Likewise, the T-forward measure is characterized by the fact that the price of any traded asset divided by bond $P(\cdot,T)$ is a martingale under it. The "things" that we divide traded assets by (or discount them with) are commonly referred to as "numeraires". The choice of numeraire is not limited to the money-market account or discount bonds. *Any traded asset with always-positive value can be used as a numeraire.*

Let N_t be a traded asset such that $N_t > 0$ for all t, $\mathbf{Q}$-almost surely. Then

$$\frac{N_t}{B_t}$$

is a martingale under $\mathbf{Q}$ (see [L1]). Therefore, we can construct a measure (see previous sections) $\hat{\mathbf{Q}}^N$ that corresponds to numeraire N in the following way (also recall Principle of Numeraire Invariance from Lecture 2 in [SC]).

Theorem 8.1 (General Change of Numeraire)**.** *There exists a measure $\hat{\mathbf{Q}}^N$, absolutely continuous with respect to $\mathbf{Q}$, such that*

- *For any traded asset A, any t, T, $t \leq T$,*
$$A_t = N_t \hat{\mathbf{E}}^N \left(N_T^{-1} A_T \middle| \mathcal{F}_t \right);$$
- *The value of A_t "discounted by" N_t,*
$$\frac{A_t}{N_t}$$
is a martingale under $\hat{\mathbf{Q}}^N$.

Proof. Take the measure $\hat{\mathbf{Q}}^N$ defined by its Radon-Nikodim derivative

$$\left. \frac{d\hat{\mathbf{Q}}^N}{d\mathbf{Q}} \right|_{\mathcal{F}_t} = \frac{N_t/B_t}{N_0/B_0}.$$

Then apply Bayes' Rule, so that

$$\begin{aligned} N_t \hat{\mathbf{E}}^N \left(N_T^{-1} A_T \middle| \mathcal{F}_t \right) &= B_t \frac{N_t/B_t}{N_0/B_0} \hat{\mathbf{E}}^N \left(B_T^{-1} \left(\frac{N_T/B_T}{N_0/B_0} \right)^{-1} A_T \middle| \mathcal{F}_t \right) \\ &= B_t \mathbf{E} \left(B_T^{-1} A_T \middle| \mathcal{F}_t \right) \\ &= A_t. \end{aligned}$$

First assertion is proven. The second one follows immediately because we have just proven that

$$\frac{A_t}{N_t} = \hat{\mathbf{E}}^N \left(\frac{A_T}{N_T} \middle| \mathcal{F}_t \right)$$

for all t, T, $t \leq T$ (which is exactly the martingale condition). ∎

Interest-rate models have traditionally been constructed in the risk-neutral measure. But the last theorem shows that there is nothing unique about the money-market account B_t, and we can use almost any traded instrument to "discount" other instruments. An arbitrage-free model can be constructed using any numeraire. *If a model is arbitrage-free in one measure, it is arbitrage-free in all measures.*

8.2. **Example of a useful numeraire.** Suppose we have a tenor structure

$$\begin{aligned} 0 &= t_0 < t_1 < \cdots < t_K, \\ \tau_i &= t_i - t_{i-1}, \end{aligned}$$

and a (forward-starting) payers swap with notional 1 and fixed rate C. Suppose we have an option on that swap (payers swaption) that expiries at time t_1. In an HJM model, the value of the swaption is given by

$$V_t = B_t \mathbf{E}\left(B_{t_1}^{-1} \left((1 - P(t_1, t_K)) - C \sum_{i=2}^{K} P(t_1, t_i)\, \tau_i \right)^+ \middle| \mathcal{F}_t \right). \tag{8.1}$$

Let us use the (normalized) value of the fixed leg as a numeraire,

$$N_t = \sum_{i=2}^{K} P(t, t_i)\, \tau_i.$$

This is definitely a traded asset for $0 \le t \le t_1$ since it is just a linear combination of bonds. Using General Change of Numeraire Theorem, we have from (8.1) that

$$\begin{aligned} V_t &= N_t \hat{\mathbf{E}}^N \left(N_{t_1}^{-1} \left((1 - P(t_1, t_K)) - C \sum_{i=2}^{K} P(t_1, t_i)\, \tau_i \right)^+ \middle| \mathcal{F}_t \right) \\ &= N_t \hat{\mathbf{E}}^N \left(\left(\frac{(1 - P(t_1, t_K)) - C \sum_{i=2}^{K} P(t_1, t_i)\, \tau_i}{\sum_{i=2}^{K} P(t_1, t_i)\, \tau_i} \right)^+ \middle| \mathcal{F}_t \right) \\ &= N_t \hat{\mathbf{E}}^N \left(\left(\frac{1 - P(t_1, t_K)}{\sum_{i=2}^{K} P(t_1, t_i)\, \tau_i} - C \right)^+ \middle| \mathcal{F}_t \right) \\ &= N_t \hat{\mathbf{E}}^N \left((F_{t_1} - C)^+ \middle| \mathcal{F}_t \right) \\ &= \left[\sum_{i=2}^{K} P(t, t_i)\, \tau_i \right] \hat{\mathbf{E}}^N \left((F_{t_1} - C)^+ \middle| \mathcal{F}_t \right), \end{aligned}$$

where F_t is a swap break-even rate,

$$F_t = \frac{1 - P(t, t_K)}{\sum_{i=2}^{K} P(t, t_i)\,\tau_i}.$$

Note how similar this formula is to Black's formula! If we can find a model such that the swap rate F_t follows geometric Brownian motion under $\hat{\mathbf{Q}}^N$, we would have a model in which this particular swaption is priced using industry-standard Black's model.

We will have more to say about this type of models later in the course.

9. Conclusions

A risk-neutral measure is characterized by money-market account being its numeraire. Almost any traded instrument can be used as a numeraire however, giving a choice of measures to use in computing values of other instruments. Valuation of many instruments can be significantly simplified by the appropriate choice of a numeraire. Of special importance are measures that correspond to using bonds as numeraires. These measures are called "forward measures". They can be conveniently characterized in the framework of Gaussian HJM models.

References

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